The Gaussian Beam

Represent the real wave form $u(r,t)$ in terms of a complex function

$$U(\bar{r},t) = a(\bar{r})\, \exp[j\,\phi(\bar{r})]\, \exp(j 2\pi f t)$$

$$u(\bar{r},t) = \mathrm{Re}\{U(\bar{r},t)\} = \tfrac{1}{2}\left[U(\bar{r},t) + U^*(\bar{r},t)\right]$$

the wave equation is:

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0$$

the time independent factor

$$A(\bar{r}) = a(\bar{r})\, \exp[j\,\phi(\bar{r})]$$ is complex amplitude

$$U(\bar{r},t) = A(\bar{r})\, \exp[j 2\pi f\, t]$$

substitute into wave equation

$$(\nabla^2 + K^2)\, U(\bar{r}) = 0$$ Helmholtz equation

$$K = \frac{2\pi f}{c}$$

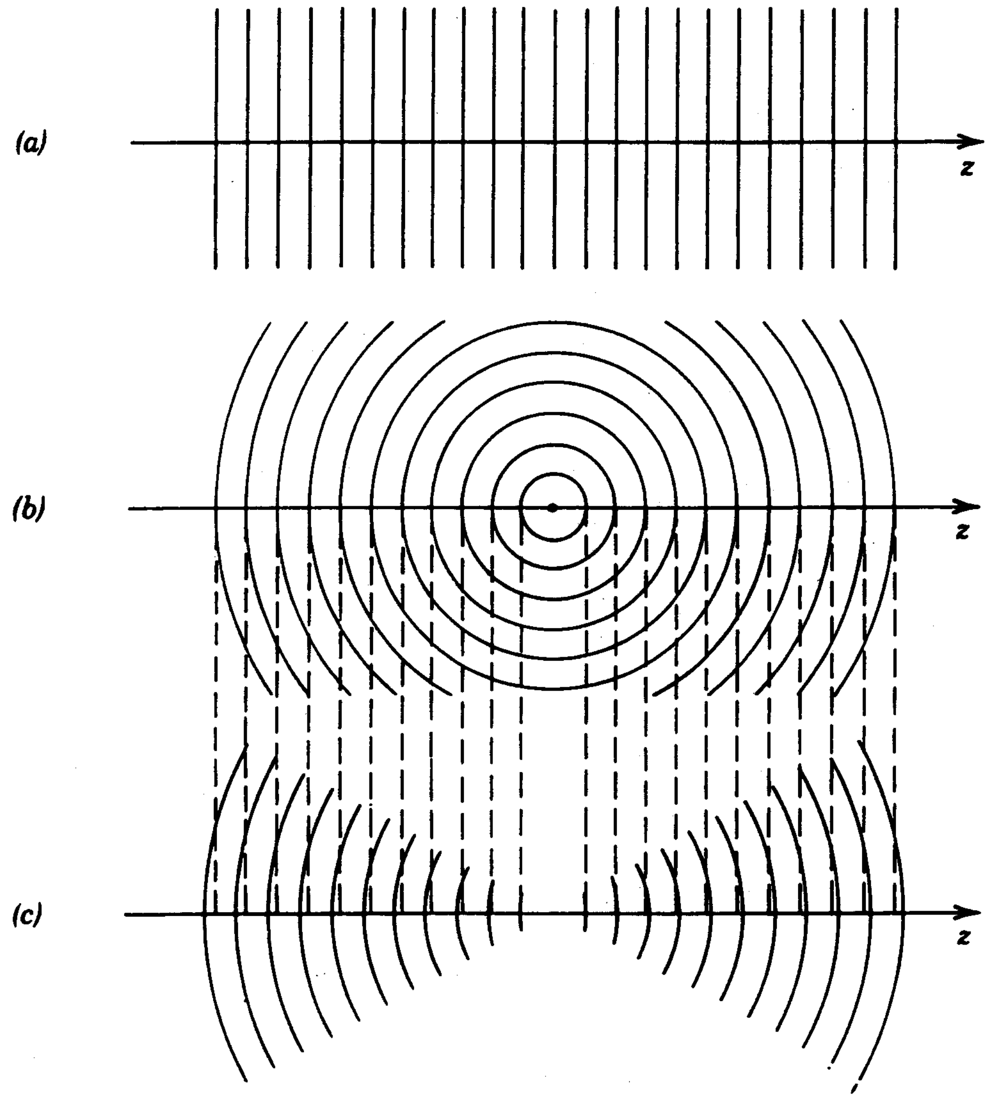


Figure 3.1-8 Wavefronts of (a) a uniform plane wave; (b) a spherical wave; (c) a Gaussian beam. At points near the beam center, the Gaussian beam resembles a plane wave. At large z the beam behaves like a spherical wave except that the phase is retarded by 90° (shown in this diagram by a quarter of the distance between two adjacent wavefronts).

There are a variety of solutions to the Helmholtz equation

Plane Wave
Spherical Wave

We are interested in an approximation known as the paraxial approximation.

The complex amplitude is slowly varying with position

$$U(\bar{r}, t) = A(\bar{r}) \exp[-jkz]$$

$$\frac{dA}{dz} \ll kA$$

$$\frac{d^2A}{dz^2} \ll k^2A$$

The Helmholtz equation becomes

$$\nabla_T^2 A - j2k\frac{dA}{dz} = 0$$

where $\nabla_T^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2}$

One solution to the paraxial Helmholtz equation is the Gaussian

$$A(\bar{r}) = \frac{A_1}{z} \exp\left[-j k \frac{\rho^2}{2z}\right], \quad \rho^2 = x^2 + y^2$$

a shifted version is also a solution

$$A(\bar{r}) = \frac{A_1}{q(z)} \exp\left[-j k \frac{\rho^2}{2q(z)}\right] \qquad q = z + j z_0$$

... z_0 is the Rayleigh range

Separate out the real and imaginary parts

$$\frac{1}{q(z)} = \frac{1}{z + j z_0} = \frac{1}{R(z)} - j \frac{\lambda}{\pi W^2(z)}$$

$$U(\bar{r}) = A_0 \frac{W_0}{W(z)} \exp\left[-\frac{\rho^2}{W^2(z)}\right] \exp\left[-jkz - jk\frac{\rho^2}{2R(z)} + j\zeta(z)\right]$$

$$W(z) = W_0\left[1 + \left(\frac{z}{z_0}\right)^2\right]^{1/2}$$

$$R(z) = z\left[1 + \left(\frac{z}{z_0}\right)^2\right]$$

$$\zeta(z) = \tan^{-1}\left(\frac{z}{z_0}\right)$$

$$W_0 = \left(\frac{\lambda z_0}{\pi}\right)^{1/2}$$

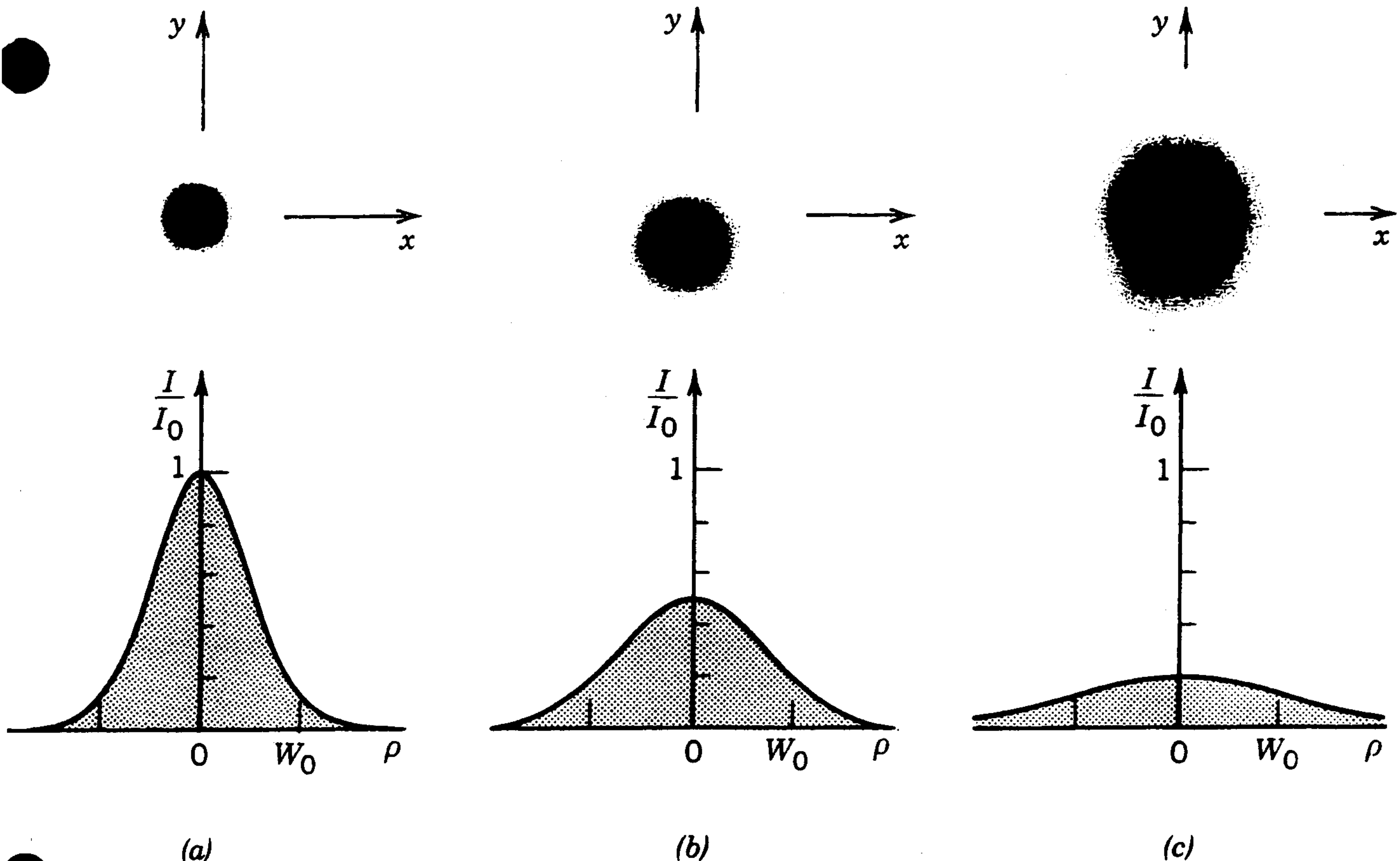
The intensity is $I(\bar{r}) = |U(\bar{r})|^2$

$$I(\rho, z) = I_0\left[\frac{W_0}{W(z)}\right]^2 \exp\left[-\frac{2\rho^2}{W^2(z)}\right]$$

The total power is: $P(z) = \int_0^\infty I(\rho, z)\, 2\pi\rho\, d\rho$

$$P = \frac{1}{2} I_0\, \pi W_0^2$$

$$I = \frac{2P}{\pi W^2(z)} \exp\left[-\frac{2\rho^2}{W^2(z)}\right]$$



ure 3.1-1 The normalized beam intensity I/I_0 as a function of the radial distance ρ at ferent axial distances: (a) $z = 0$; (b) $z = z_0$; (c) $z = 2z_0$.

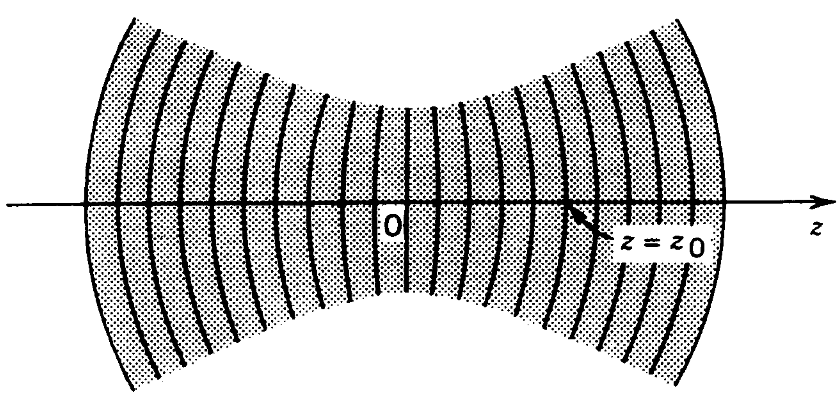


Figure 3.1-7 Wavefronts of a Gaussian beam.

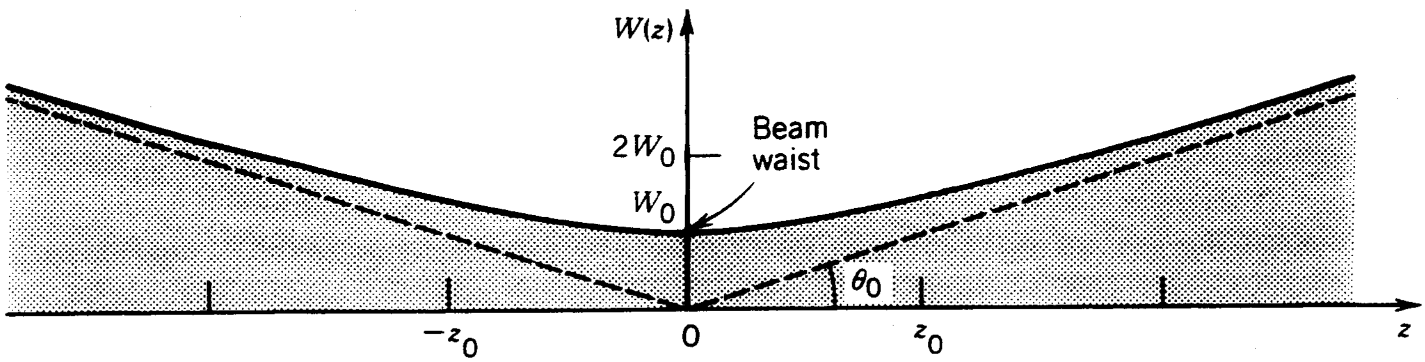


Figure 3.1-3 The beam radius $W(z)$ has its minimum value W_0 at the waist ($z = 0$), reache $\sqrt{2}\,W_0$ at $z = \pm z_0$, and increases linearly with z for large z.

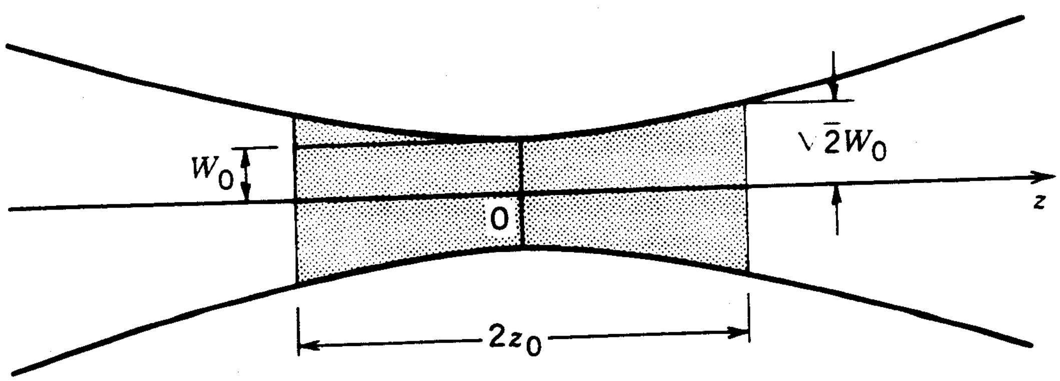


Figure 3.1-4 The depth of focus of a Gaussian beam.

when $\rho = W$ $\quad I = I_0 \, e^{-2}$

W : is the beam waist defined as the $1/e^2$ point

The width increases with distance z where the normalized parameter z_0 is called the Rayleigh range

As $z \gg z_0$ the beam achieves a linear divergence given by

$$\theta = \frac{\lambda}{\pi W_0}$$

Divergence depends on : Wavelength
beam waist

Propagation through a thin lens

multiply the complex amplitude of the Gaussian by the complex transmittance of the thin lens

$$\exp\left[\frac{i k \rho^2}{2f}\right]$$

The phase is altered without changing the Gaussian waist