The Gaussian Beam

Represent the real waveform \( u(r, t) \) in terms of a complex function

\[
U(r, t) = a(r) \exp[i \phi(r)] \exp[i 2\pi ft]
\]

\[
u(r, t) = \text{Re} \{ U(r, t) \} = \frac{1}{2} [ u(r, t) + u^*(r, t) ]
\]

the wave equation is:

\[
\nabla^2 U - \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = 0
\]

the time independent factor

\[
A(r) = a(r) \exp[i \phi(r)] \text{ is complex amplitude}
\]

\[
u(r, t) = A(r) \exp[i 2\pi ft]
\]

substitute into wave equation

\[
(\nabla^2 + k^2) u(r) = 0 \quad \text{Helmholtz equation}
\]

\[
k = \frac{2\pi f}{c}
\]
Figure 3.1-8 Wavefronts of (a) a uniform plane wave; (b) a spherical wave; (c) a Gaussian beam. At points near the beam center, the Gaussian beam resembles a plane wave. At large $z$ the beam behaves like a spherical wave except that the phase is retarded by $90^\circ$ (shown in this diagram by a quarter of the distance between two adjacent wavefronts).
There are a variety of solutions to the Helmholtz equation:

- Plane Wave
- Spherical Wave

We are interested in an approximation known as the paraxial approximation. The complex amplitude is slowly varying with position:

$$ U(r, \theta) = A(r) \exp[-i k z] $$

$$ \frac{dA}{dz} \ll kA $$

$$ \frac{d^2A}{dz^2} \ll k^2A $$

The Helmholtz equation becomes

$$ \nabla^2 A - \frac{1}{r} \frac{d}{dr} \frac{dA}{dr} = 0 $$

where

$$ \nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} $$
One solution to the paraxial Helmholtz equation is the Gaussian

\[ A(\vec{r}) = \frac{A_i}{z} \exp \left[ -ik \frac{\rho^2}{2z} \right] \quad \rho^2 = x^2 + y^2 \]

A shifted version is also a solution

\[ A(\vec{r}) = \frac{A_i}{q(z)} \exp \left[ -ik \frac{\rho^2}{2q(z)} \right] \quad q = z + g z_0 \]

\( z_0 \) is the Rayleigh range

Separate out the real and imaginary parts

\[ \frac{1}{q(z)} = \frac{1}{z + g z_0} = \frac{1}{R(z)} - g \frac{\lambda}{\pi W^2(z)} \]
\[ U(r) = A_0 \frac{W_0}{W(z)} \exp \left[ -\frac{\rho^2}{W(z)} \right] \exp \left[ -\frac{\alpha k z}{2} - \frac{\alpha k \rho^2}{2R(z)} + \frac{1}{2} \xi(z) \right] \]

\[ W(z) = W_0 \left[ 1 + \left( \frac{z}{z_0} \right)^2 \right]^{1/2} \]

\[ R(z) = z \left[ 1 + \left( \frac{z}{z_0} \right)^2 \right] \]

\[ \xi(z) = \tan^{-1} \left( \frac{z}{z_0} \right) \]

\[ W_0 = \left( \frac{\lambda z_0}{\pi} \right)^{1/2} \]

The intensity is 
\[ I(r) = |U(r)|^2 \]

\[ I(\rho, z) = J_0 \left[ \frac{W_0}{W(z)} \right]^2 \exp \left[ - \frac{2\rho^2}{W(z)} \right] \]

The total power is:
\[ P(z) = \int_0^\infty I(\rho, z) 2\pi \rho \, d\rho \]

\[ P = \frac{1}{2} J_0 \pi W_0^{\alpha} \]

\[ I = \frac{2P}{\pi W^2(z)} \exp \left[ - \frac{2\rho^2}{W(z)} \right] \]
Figure 3.1-1 The normalized beam intensity $I/I_0$ as a function of the radial distance $\rho$ at different axial distances: (a) $z = 0$; (b) $z = z_0$; (c) $z = 2z_0$.

Figure 3.1-7 Wavefronts of a Gaussian beam.
Figure 3.1-3  The beam radius $W(z)$ has its minimum value $W_0$ at the waist ($z = 0$), reaches $\sqrt{2} W_0$ at $z = \pm z_0$, and increases linearly with $z$ for large $z$.

Figure 3.1-4  The depth of focus of a Gaussian beam.
when \( p = W \)

\[ I = I_0 e^{-2} \]

\( W \) is the beam waist defined as the \( 1/e^2 \) point.

The width increases with distance \( Z \) where the normalized parameter \( Z_0 \) is called the Rayleigh range.

As \( Z > Z_0 \), the beam achieves a linear divergence given by

\[ \Theta = \frac{\lambda}{\pi W_0} \]

Divergence depends on: Wavelength, beam waist.

Propagation through a thin lens

multiply the complex amplitude of the Gaussian by the complex transmittance of the thin lens

\[ \exp \left[ \frac{i k p^2}{2f} \right] \]

The phase is altered without changing the Gaussian waist.