Each component of the electric and magnetic field must satisfy the wave equation

$$\nabla^2 U + n^2 k^2 U = 0$$

where $n = n_1$ in the core $(r < a)$
$n = n_2$ in the cladding $(r > a)$

The wave is a traveling wave in the z direction
U must be a periodic function of the angle ϕ with period 2π

$$U(r, \phi, z) = u(r)\, e^{-jl\phi} e^{-j\beta_z z}$$

The wave equation in cylindrical coordinates

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} + n^2 k^2 U = 0$$

becomes

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(n^2 k^2 - \beta_z^2 - \frac{l^2}{r^2}\right) u = 0$$

since $n_2 k < \beta_z < n_1 k$
we define

$$K^2 = (n_1 k)^2 - \beta^2$$

$$\gamma^2 = \beta^2 - (n_2 k)^2$$

The wave equation becomes

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left[K^2 - \left(\frac{l}{r}\right)^2\right] u = 0 \qquad r < a$$

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \left[\gamma^2 + \left(\frac{l}{r}\right)^2\right] u = 0 \qquad r > a$$

$$u(r) \propto \begin{cases} J_l(Kr) & r < a \\ K_l(\gamma r) & r > a \end{cases}$$

and

$$K^2 + \gamma^2 = (n_1^2 - n_2^2)\, k$$

Assuming weak modulation $\sqrt{n_1^2 - n_2^2} << 1$
Applying boundary conditions

$$(\kappa a)\,\frac{J_\ell'(\kappa a)}{J_\ell(\kappa a)} = (\gamma a)\,\frac{K_\ell'(\gamma a)}{K_\ell(\gamma a)}$$

$$J_\ell'(x) = \pm J_{\ell\pm 1} \pm \ell\,\frac{J_\ell(x)}{x}$$

$$K_\ell'(x) = -K_{\ell\pm 1}(x) \mp \ell\,\frac{K_\ell(x)}{x}$$

$$\frac{(\kappa a)}{J_\ell(\kappa a)}\left[\pm J_{\ell\pm 1}(\kappa a) \mp \ell\,\frac{J_\ell(\kappa a)}{\kappa a}\right] = \frac{\gamma a}{K_\ell(\gamma a)}\left[-K_{\ell\pm 1}(\gamma a) \mp \ell\,\frac{K_\ell(\gamma a)}{\gamma a}\right]$$

$$\pm(\kappa a)\,\frac{J_{\ell\pm 1}(\kappa a)}{J_\ell(\kappa a)} \cancel{\mp\ell} = -\gamma a\,\frac{K_{\ell\pm 1}(\gamma a)}{K_\ell(\gamma a)} \cancel{\mp\ell}$$

$$\pm \kappa a\,\frac{J_{\ell\pm 1}(\kappa a)}{J_\ell(\kappa a)} = -\gamma a\,\frac{K_{\ell\pm 1}(\gamma a)}{K_\ell(\gamma a)}$$

Remember that κ and γ are related

$$\kappa^2 + \gamma^2 = (n_1^2 k^2 - \beta^2) + \beta^2 - (n_2 k)^2$$

$$= k^2 (n_1^2 - n_2^2)$$

$$\sqrt{(\kappa a)^2 + (\gamma a)^2} = \frac{2\pi}{\lambda}\, a\,\sqrt{n_1^2 - n_2^2}$$

$$V = \frac{2\pi}{\lambda}\, a\,\sqrt{n_1^2 - n_2^2}$$

$$\gamma a = \sqrt{V^2 - (\kappa a)^2}$$

so $\gamma a \to 0$
as $\kappa a \to V$

The left hand side $-\gamma a\,\dfrac{K_{\ell\pm 1}(\gamma a)}{K_\ell(\gamma a)} \to 0$ as $\kappa a \to V$

The supported modes are the ones in which the zeros of $J_{\ell\pm 1}(x) < V$

$J_1(x) \to 0$ as $x \to 0$

but

$J_0(x) \to 1$ as $x \to 0$

So $x \dfrac{J_1(x)}{J_0(x)} \to 0$ as $x \to 0$

For all other modes we find where

$$J_{\ell \pm 1}(x) < V$$

ZEROS

$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$
2.405				
	3.832			
		5.136		
5.52				
			6.38	
	7.016			
				7.588
		8.417		
8.634				
			9.761	

The most important quantity is what is the single mode operation range.

This is when

$$V < 2.405$$

OR

$$2\pi \left(\frac{\lambda}{a}\right) \sqrt{n_1^2 - n_2^2} < 2.405$$