EE466 Optical Engineering
Homework #5

1. Derive the numerical aperture of a fiber, which is defined as \( NA = \sin(\theta) \), where \( \theta \) is the maximum ray angle that can be confined to the fiber.

2. An LED has an intensity pattern given by \( I(\theta) = \frac{1}{\pi r^2} P_o \cos(\theta) \). The LED is placed against the end of a large core optical fiber.
   a. Show that the amount of power coupled into an optical fiber is \( P = (NA)^2 P_o \).
      Assume that all light with an angle less than the maximum acceptance angle of the fiber is coupled into the fiber.
   b. Why must the core be large for the assumption in (a) to be valid?

3. Derive the \( J_{l+\pm}(k_r a) \frac{J_{l+\pm}(k_r a)}{J_l(k_r a)} = \pm \gamma a \frac{K_{l+\pm}(\gamma a)}{K_l(\gamma a)} \) from \( k_r a \frac{J'_{l+}(k_r a)}{J_l(k_r a)} = \gamma a \frac{K'_{l+}(\gamma a)}{K_l(\gamma a)} \).

4. (Book problem 9.2-1) A step-index optical fiber has radius \( a = 5 \text{ \mu m} \), core refractive index \( n_f = 1.45 \), \( \Delta = 0.002 \). Determine the shortest wavelength \( \lambda_c \) for which the fiber is a single-mode waveguide. If the wavelength is changed to \( \lambda_c / 2 \), identify the indices \((l, m)\) of all the modes.

5. (Book problem 9.3-3) At \( \lambda_o = 820 \text{ nm} \) the absorption loss of a fiber is 0.25 dB/km and the scattering loss is 2.25 dB/km. If the fiber is used instead at \( \lambda_o = 600 \text{ nm} \), and measurements of the heat generated by light absorption gives a loss of 2 dB/km, estimate the total attenuation at \( \lambda_o = 600 \text{nm} \).

6. The power measured out of an arbitrary length of optical fiber is \( P = 100 \mu W \). A 10m section of optical fiber is then cut off. The power is then measured to be 105\( \mu W \). What is the attenuation of the optical fiber in dB/km? This method used to measure the fiber attenuation is called the cut-back method and removes the insertion loss of the source from the measurement.

7. Standard single mode optical fiber has an attenuation of \( \alpha = 0.2 \text{ dB/km} \). The laser has a power of \( P = 5 \text{ mW} \). The coupling between the laser and the fiber is 70%. The coupling between the fiber and the detector is 90%. The fiber comes in 10km spools. Each splice between fiber spools results in a 0.3 dB loss. What is the maximum link distance for which the received power is greater than -22 dBm?
1. Derive the numerical aperture equation of an optical fiber.

\[
\begin{align*}
\theta_c &= \sin^{-1}\left(\frac{\theta_i}{n_2}\right) \\
\sin \theta_A &= n_1 \sin \left[90 - \sin^{-1}\left(\frac{n_2}{n_1}\right)\right] \\
\sin \theta_A &= n_1 \left[\sin \theta_i \cos \sin^{-1}\left(\frac{n_2}{n_1}\right) - \cos \theta_i \sin \sin^{-1}\left(\frac{n_2}{n_1}\right)\right] \\
\sin \theta_A &= n_1 \cos \sin^{-1}\left(\frac{n_2}{n_1}\right) \\
&= n_1 \left(\sqrt{n_2^2 - n_1^2}\right) \\
\sin \theta_A &= \sqrt{n_2^2 - n_1^2}
\end{align*}
\]

2. The power measured out of an arbitrary length of optical fiber is \( P = 100 \, \text{mW} \). A 10 m section of optical fiber is then cut off. The power is then measured to be \( 105 \, \text{mW} \). What is the attenuation of the optical fiber?

\[
\begin{align*}
P_1 &= 105 \, \text{mW} = -9.788 \, \text{dBm} \\
P_2 &= 100 \, \text{mW} = -10 \, \text{dBm} \\
P_1 - P_2 &= -9.788 + 10 = 0.212 \, \text{dB in 10m} \\
\alpha_{dB} &= \frac{0.212 \, \text{dB}}{10 \, \text{m}} \\
\alpha_{dB} &= 21.2 \, \text{dB/km}
\end{align*}
\]
(2) \[ I(\theta) = \frac{1}{\pi \theta^2} P_0 \cos \theta \]

The collection of the optical fiber is \( \Theta < \sin^{-1}(NA) \)

Use a spherical integral to calculate the total power emitted into the fiber acceptance cone.

\[
\text{Power} = \int_0^{\sin^{-1}(NA)} \int_0^{2\pi} \frac{1}{\pi \theta^2} P_0 \cos \theta \rho^2 \sin \theta \, d\theta \, d\phi
\]

\[
= \left( 2\pi \right) \left( \frac{P_0}{\pi} \right) \int_0^{\sin^{-1}(NA)} \frac{1}{\theta^2} \rho^2 \sin \theta \, d\theta
\]

\[
= \left( \frac{P_0}{2} \right) \int_0^{\sin^{-1}(NA)} \sin^4 \theta \, d\theta
\]

\[
= \left( \frac{P_0}{2} \right) \left[ -\cos(2\theta) \right]_0^{\sin^{-1}(NA)}
\]

\[
= \frac{P_0}{2} \left[ 2 \sin^2 \Theta - 1 \right]_0^{\sin^{-1}(NA)}
\]

\[
= \frac{P_0}{2} \left[ 2 \left( \sin \left( \sin^{-1}(NA) \right) \right)^2 - 2 \sin^2(\Theta) \right]
\]

\[
= \frac{P_0}{2} \left( \sin^2(NA) \right)^2 - 2 \sin^2(\Theta)
\]

\[
= P_0 (NA)^2
\]
(3) \[ K_T (a) \frac{J_e'(K_T, a)}{J_e(K_T, a)} = f_a \frac{K_e'(f_a)}{K_e(f_a)} \]

Use the relationships: \[ J_e'(x) = \frac{1}{x} J_e(x) + \ell \frac{J_e(x)}{x} \]

\[ K_e'(x) = -K_{e1}(x) - \ell \frac{K_e(x)}{x} \]

\[ K_T (a) \left( \frac{\pm J_e (K_T, a)}{J_e(K_T, a)} + \ell \frac{J_e(K_T, a)}{K_T} \right) = f_a \left( -K_{e1}(f_a) - \ell \frac{K_e(f_a)}{f_a} \right) \]

\[ K_T (a) \left( \frac{\pm J_e (K_T, a)}{J_e(K_T, a)} + \ell \frac{J_e(K_T, a)}{K_T} \right) = f_a \left( -K_{e1}(f_a) - \ell \frac{K_e(f_a)}{f_a} \right) \]

\[ K_T (a) \left( \frac{\pm J_e (K_T, a)}{J_e(K_T, a)} + \ell \frac{J_e(K_T, a)}{K_T} \right) = f_a \left( -K_{e1}(f_a) - \ell \frac{K_e(f_a)}{f_a} \right) \]

\[ K_T (a) \frac{J_e (K_T, a)}{J_e(K_T, a)} = \pm f_a \frac{K_{e1}(f_a)}{K_e(f_a)} \]
(4) \[ \bar{q} = 10 \]
\[ \bar{\tau}_0 = 1.45 \]
\[ \Delta = 0.01 \]

\[ M = \frac{\bar{q}}{\bar{\tau}_0^2} \left( \frac{\bar{\tau}_0^2}{\bar{\tau}_0} \right) \]
\[ = \frac{10}{1.45^2} \left( \frac{2 \pi a}{\bar{\tau}_0} \right)^2 \cdot 1.45 \sqrt{\Delta} \]
\[ = \frac{10}{1.45^2} \left( \frac{2 \pi \bar{\tau}_0^2}{\bar{\tau}_0} \right) \cdot \left( 1.45 \right) \cdot 0.01 \]
\[ M = \left( \frac{10}{1.45^2} \right) (809.55) \]

<table>
<thead>
<tr>
<th>( \bar{\tau} )</th>
<th>( M )</th>
</tr>
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<tr>
<td>1.9</td>
<td>394</td>
</tr>
<tr>
<td>2.0</td>
<td>404</td>
</tr>
<tr>
<td>2.1</td>
<td>414</td>
</tr>
<tr>
<td>( \infty )</td>
<td>809</td>
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</tbody>
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\[ \frac{\sigma_T}{\bar{\tau}} = \frac{\Delta^2}{\bar{\tau}} = \frac{(0.01)^2}{\left( \frac{2 \pi \bar{\tau}_0^2}{\bar{\tau}_0} \right)} \]
\[ = \frac{1.208 \times 10^{-13}}{ \text{m} } = 121 \text{ Pa} \text{m} \]
A step-index fiber has radius \( a = 5 \text{mm} \), core refractive index \( n_1 = 1.45 \), and fractional refractive-index change \( \Delta = 0.002 \). Determine the shortest wavelength \( \lambda_c \) for which the fiber is a single-mode waveguide. If the wavelength is changed to \( \lambda / 2 \), identify the indices \((l,m)\) of all the guided modes.

\[
\sqrt{\frac{2 \pi}{\lambda}} a \sqrt{2 \Delta} \leq 2.405
\]

\[
\lambda_c = \frac{2 \pi}{2.405} (5 \text{mm})(1.45) \sqrt{(2)(0.002)}
\]

\[\lambda_c = 1.197 \text{ mm}\]

The normalized frequency changes to

\[
\sqrt{\frac{2 \pi}{\lambda}} = (2)(2.405) = 4.81
\]

The LP modes with \( 2 \pi a \Delta \) less than 4.81 are

\[l = 0, m = 2 \Rightarrow 3.832\]
\[l = 1, m = 1 \Rightarrow 2.405\]

The modes are: \( \text{LP}_{02}, \text{LP}_{01}, \text{LP}_{11} \)

At \( \lambda_0 = 820 \text{ nm} \) the absorption loss of a fiber is 0.25 dB/km and the scattering loss is 2.25 dB/km. If the fiber is used at \( \lambda_0 = 630 \text{ nm} \) and the absorption is 2 dB/km, estimate the total attenuation at \( \lambda = 600 \text{ nm} \)

\[
\alpha_{\text{scatt}, 600} = 2.25 \text{ dB/km}
\]

\[
\alpha_{\text{scatter}} = (0.25)(2.45) = 0.5175 \text{ km}^{-1} = \frac{CR}{\lambda} = \frac{CR}{(520 \times 10^{-9})^4}
\]

\[
CR = 0.5175 \times (0.33 \times 10^{-3})^3 = 2.34 \times 10^{-19} \text{ m}^3
\]

\[
\alpha_{\text{scatter}} = \frac{2.34 \times 10^{-19}}{(600 \times 10^{-9})^4} = 180.6 \text{ m}^{-1} = 0.23 \text{ dB/km (180.6 km)}
\]

\[\alpha_{\text{scatt}, 630} = 7.95 \text{ dB/km}\]

\[\alpha = 2.25 \text{ dB/km} + 7.95 \text{ dB/km}\]

\[\alpha = 10.25 \text{ dB/km}\]
Standard single mode optical fiber has an attenuation of \( \alpha = 0.2 \) dB/km. The laser has a power of \( P_t = 5 \) mW. The coupling between the laser and the fiber is 70%. The coupling between the fiber and the detector is 90%. The fiber comes in 10 km spools. Each splice between fiber spools results in a 0.3 dB loss. What is the maximum link distance for which the received power is greater than -22 dBm.

\[
P_t = 5 \text{ mW} = 7 \text{ dBm} \quad \text{(transmitted power)}
\]
\[
P_r = -22 \text{ dBm}
\]

\[
\begin{align*}
\text{Fiber Connection} & : 0.7 = 10 \log_{10} (0.7) = -1.55 \text{ dB} \\
\text{Detector Connection} & : 0.9 = 10 \log_{10} (0.9) = -0.46 \text{ dB} \\
\text{Fiber loss} & : \alpha L = (0.2)(10) = -2 \text{ dB} \\
\text{Fiber splice} & : -0.3 \text{ dB}
\end{align*}
\]

Power Budget:
\[
(P_r - P_t) = -1.55 + N (-2 - 0.3) - 0.46 \\
-2.9 + 1.55 + 0.46 = N (-2.3) \\
N = 11.7
\]

\[
\frac{N}{11} = 11
\]

Loss:
\[
1.55 + (11)(0.2) + 11(0.3) + 0.46 = 27.31 \text{ dB}
\]

Remaining budget is:
\[
29 - 27.31 = 1.69
\]

Remaining fiber length is:
\[
L = \frac{1.69}{0.2} = 8.45 \text{ km}
\]

Total length:
\[
L = 118.45 \text{ km}
\]